SOLUTION EXERCISE SHEET 24

Exercise 1. A direct computation shows that

$$A\vec{w} = \begin{pmatrix} (\partial_1 f_2 - \partial_2 f_1)w_2 + (\partial_1 f_3 - \partial_3 f_1)w_3 \\ (\partial_2 f_1 - \partial_1 f_2)w_1 + (\partial_2 f_3 - \partial_3 f_2)w_3 \\ (\partial_3 f_1 - \partial_1 f_3)w_1 + (\partial_3 f_2 - \partial_2 f_3)w_2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \vec{v}^T A\vec{w} &= (\partial_1 f_2 - \partial_2 f_1)w_2 v_1 + (\partial_1 f_3 - \partial_3 f_1)w_3 v_1 \\ &\quad + (\partial_2 f_1 - \partial_1 f_2)w_1 v_2 + (\partial_2 f_3 - \partial_3 f_2)w_3 v_2 \\ &\quad + (\partial_3 f_1 - \partial_1 f_3)w_1 v_3 + (\partial_3 f_2 - \partial_2 f_3)w_2 v_3 \\ &= \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix} \cdot \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \\ &= \operatorname{rot}\vec{f} \cdot (\vec{v} \times \vec{w}). \end{aligned}$$

Exercise 2. Note that, as by assumption $\overline{V_1} \subset U$, we have that

$$\partial(U \setminus \overline{V_1}) = \partial U \cup \partial V_1.$$

Furthermore, by the compactness of $\overline{V}_1$ we know that $\operatorname{dist}(\overline{V_1}, \partial U) = c > 0$. Consequently, we can cover $\overline{U}\setminus V_1$ by finitely many open balls of the form $B_{\frac{c}{6}}(x_1), \dots, B_{\frac{c}{6}}(x_n)$. Next, for $i = 1, \dots n$ we set $B_i = B_{\frac{c}{3}}(x_i)$. Now, for each of the B_i exactly one of the following 3 options holds

(1) $B_i \cap (\partial U \cup \partial V_1) = \emptyset$,
(2) $B_i \cap \partial U \neq \emptyset$ and $B_i \cap \partial V_1 = \emptyset$,
(3) $B_i \cap \partial V_1 \neq \emptyset$ and $B_i \cap \partial U = \emptyset$.

For $i = 1, \dots n$ we now let $\{\psi_i\}_{i=1}^n$ be a smooth partition of unity such that the support of each ψ_i is a compact subset of B_i. Then

$$\iiint_{\overline{U}\setminus V_1} \operatorname{div}\vec{f} = \sum_{i=1}^n \iiint_{(\overline{U}\setminus V_1)\cap B_i)} \operatorname{div}(\psi_i \vec{f}).$$

Now we make a case distinction according to the three options listed above. If B_i is such that (1) holds, then, by the divergence theorem and the fact that ψ_i has compact support in B_i, it follows that

$$\iiint_{(\overline{U}\setminus V_1)\cap B_i)} \operatorname{div}(\psi_i \vec{f}) = 0.$$

In case (2) the divergence Theorem yields

$$\iiint_{(\overline{U}\setminus V_1)\cap B_i} \operatorname{div}(\psi_i \vec{f}) = \iint_{\partial U \cap B_i} \psi_i \vec{f} \cdot \eta$$

while

$$\iiint_{(\overline{U}\setminus V_1)\cap B_i} \operatorname{div}(\psi_i \vec{f}) = \iint_{\partial V_1 \cap B_i} \psi_i \vec{f} \cdot \eta$$

in case (3). Finally, summing over i yields the claim.

Exercise 3. As suggested, we change variables to obtain that

$$\iint_{\partial B_r(0)} f(x) d\sigma = r^2 \iint_{\partial B_1(0)} f(r\omega) d\omega.$$

Then, one readily computes that

$$4\pi g'(r) = \iint_{\partial B_1(0)} \nabla f(r\omega) \cdot \omega d\omega.$$

Thus, as ω is the outgoing unit normal of the sphere, an application of the Gauss-Ostrogradskii theorem yields

$$4\pi g'(r) = r^3 \iiint_{B_1(0)} \nabla_x \cdot (\nabla f(rx))\, dx = r^3 \iiint_{B_1(0)} (\Delta f)(rx) dx = 0$$

since f is harmonic. So, $g' = 0$ and the claim follows.